RADIATIVE—CONDUCTIVE TRANSMISSION OF HEAT THROUGH OPTICALLY DENSE MEDIA

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The problem of heat transmission by conduction and radiation through a semiinfinite optically dense medium is analyzed, with incident external radiation and with convective heat transfer taken into account. An expression for the radiant thermal flux is derived from the solution to the equation of radiation flux propagation by the method of associative asymptotic expansions. The effect of the temperature gradient at the surface on the emissivity of the body is established for the medium range of absorptivity values.

In an analysis of the radiative—conductive heat transmission through optically dense media, the Rosseland approximation usually serves as the expression for the radiant component of the thermal flux. This approximation has been derived from a parametric expansion of the solution to the equation of radiation propagation [1].

An expansion in terms of the parameter results in the elimination of the arbitrary constant from the solution to the equation of radiative heat transmission and renders it unsuitable for the inhomogeneous region adjoining the boundary, where a boundary condition must be satisfied. Obviously, such an expansion cannot be considered equally useful over the entire range of the problem.

The existence of an inhomogeneous region is, as a rule, related to the expansion of the solution in terms of the parameter by which the first derivative is multiplied [2], and in the equation of radiation propagation through optically dense media such an expansion parameter is $\varepsilon = 1/k$.

To arrive at a universally useful parametric expansion in this situation is the object in the problem of particular (singular) perturbations. A universal solution is most often obtained by constructing an approximation which is uniformly close within the inhomogeneous region and then associating it with a straight parametric expansion by the method of associative asymptotic expansions.

Several interesting facts about radiative—conductive heat transmission through optically dense media can be revealed, if the method of associative expansions is applied to the solution of such problems.

We will consider the following problem:

It is desired to determine the steady-state temperature field and the thermal fluxes in a semiinfinite solid body on whose surface impinges external radiation uniformly from all directions. Through the same surface heat is transferred from that body to the ambient medium by convection.

The problem is solved under the following assumptions: 1) the boundary between the solid body and the adjoining medium is transparent to the external radiation and is also diffusive, 2) the absorptivity of the solid material does not vary with the radiation frequency, 3) the hypothesis of local dynamic equilibrium applies to the radiation, 4) the physical parameters of the material are not temperature-dependent, 5) the temperature field is uniform, and 6) the refractive indices of the solid material and the adjoining medium are both equal to unity.

We will resolve the radiation intensity in the solid into two components in opposite directions: I^+ and I^- (see Fig. 1).

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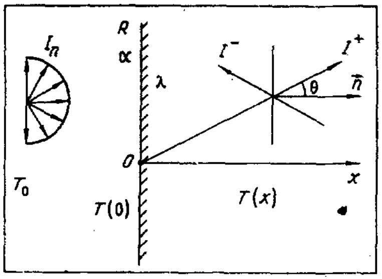

Fig. 1. Schematic diagram of the problem.

Both intensity fields are described by the equation

$$\varepsilon m \frac{dI}{dx} = B(x) - I. \tag{1}$$

In order to find the I^+ intensity distribution in the solid, we let $I = I^+$ and $m \geq 0$ in Eq. (1).

The conditions of radiation at the boundary will be stated as

$$I^+ = I^+(0) \text{ when } x = 0. \tag{2}$$

The quantity $I^+(0)$ will be determined later.

The outer expansion of the solution will be expressed in terms of the following asymptotic sequence

$$I^+ = I_0^+ + \varepsilon I_1^+ + \varepsilon^2 I_2^+ + \cdots \tag{3}$$

The unknown functions are determined by inserting this expansion into Eq. (1) and subsequently equating the respective equal-power terms in ε. As a result, we obtain

$$I^+ = B(x) - \varepsilon m B'(x) + O(\varepsilon^2). \tag{4}$$

The prime sign denotes a derivative with respect to x.

Expression (4) is not a uniform approximation to the solution to Eq. (1), because it does not match the boundary condition (2). This expansion is, therefore, unsuitable for the inhomogeneous region adjoining the boundary.

We will now examine the solution in the inhomogeneous region. For this purpose we introduce a new variable and a new function defined by the equalities:

$$X = \frac{x}{\varepsilon}; \; I^+(x, \varepsilon) = J^+(\varepsilon, X).$$

With the aid of these expressions we transform the original equation (1):

$$m \frac{dJ^+}{dX} = B(\varepsilon X) - J^+ \tag{4'}$$

and the boundary condition (2). The latter is now written as

$$J^+ = I^+(0) \text{ when } X = 0. \tag{5}$$

The transformations have resulted in the elimination of the small parameter by which the first derivative is multiplied. An expansion of the solution to Eq. (4) in terms of ε will make it possible to retain several essential properties of the solution which are lost in the outer expansion (3).

We will now represent the inner expansion by such an asymptotic sequence

$$J^+ = J_0^+ + \varepsilon J_1^+ + \varepsilon^2 J_2^+ + \cdots \tag{6}$$

The unknown functions in expansion (6) are determined by inserting (6) into (4) and (5). It must be considered here that

$$B(x) = B(\varepsilon X) = B(0) + \varepsilon B'(0) X + \cdots \tag{7}$$

It is easy to see that the unknown functions in expansion (6) are the solution to first-order differential equations with boundary conditions defined according to (5).

The binomial inner expansion is

$$J^+ = B(0) + [I^+(0) - B(0)] \exp\left(-\frac{X}{m}\right)$$
$$+ \varepsilon m B'(0) \left[\frac{X}{m} - 1 + \exp\left(-\frac{X}{m}\right)\right] + O(\varepsilon^2). \tag{8}$$

This expansion describes the solution to Eq. (1) where expansion (3) is useless.

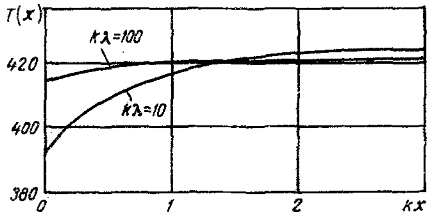

Fig. 2. Temperature field in a semiinfinite body with $\sigma = 5.67 \cdot 10^{-8}$ $W/m^2 \cdot (°K)^4$, $\alpha = 10\ W/m^2 \cdot °C$, R = 0.5, $I_n = 1300\ W/m^2$, $T_0 = 300°K$.

By combining expansions (3) and (6), one can construct a uniformly approximate solution to Eq. (1) everywhere within the range of the problem. The sought composite expansion obtained by the method of addition [2] is

$$I^+ = B(x) + [I^+(0) - B(0)] \exp\left(-\frac{x}{\varepsilon m}\right) - \varepsilon m \left[B'(x) - B'(0) \exp\left(-\frac{x}{\varepsilon m}\right)\right] + O(\varepsilon^2). \tag{9}$$

A uniform approximation for I^- is constructed in an analogous manner. The inhomogeneous region of intensity I^- lies at infinity and, therefore, the outer expansion alone will suffice for I^-, namely

$$I^- = B(x) - \varepsilon m B'(x) + O(\varepsilon^2). \tag{10}$$

The radiant flux in the solid is

$$q_r = 2\pi \int_0^1 m I^+ dm - 2\pi \int_0^{-1} m I^- dm. \tag{11}$$

With the aid of (8) and (10), this expression can be expanded as follows

$$q_r = 2\pi [I^+(0) - B(0)] E_3(kx) - \frac{4\pi}{3k} B'(x) + \frac{2\pi}{k} B'(0) E_4(kx). \tag{12}$$

Here $E_3(kx)$ and $E_4(kx)$ are exponential integrals whose properties and values are given in [3].

We note that the use of the outer expansions (3) and (10) in (11) yields, after integration, the well known Rosseland approximation for the radiation flux through optically dense media:

$$q_r = -\frac{4\pi}{3k} B'(x). \tag{13}$$

The relation derived here will be inoperative where the outer expansions (3) and (10) are unsuitable, i.e., within the region adjoining the boundary. Expression (12) yields an estimate of the error incurred by the often used Rosseland approximation.

Through media with a high absorptivity (metals, for example), therefore, the radiative transmission of heat is appreciable only within small regions near the boundary. According to expression (13), on the other hand, radiative heat transmission occurs almost nowhere.

The magnitude of $I^+(0)$ in (12) is found from the balance of radiant energy at the solid surface:

$$\pi I^+(0) = \pi(1 - R) I_n + R q^-(0). \tag{14}$$

Expanding the respective terms here yields

$$\pi I^+(0) = \pi(1 - R) I_n + \pi R B(0) + \frac{2\pi}{3k} R B'(0). \tag{15}$$

With the aid of this equality, we obtain from (12) an expression for the radiation flux at the solid surface:

$$q_r(0) = \pi(1 - R)\left[I_n - B(0) - \frac{2}{3k} B'(0)\right]. \tag{16}$$

From here it is easy to determine the intrinsic radiation from the body into the adjoining space, if one assumes that $I_n = 0$. In media with a high absorptivity the intrinsic radiation is proportional to the transparency of the boundary between the body and the adjoining medium, defined by $1 - R$, and to the Planck function. At moderate values of k, an appreciable effect on the intrinsic radiation has the temperature field gradient at the boundary.

The Fourier hypothesis and expression (12) for the radiation flux has yielded an equation for one-dimensional radiative heat transmission and conduction

$$\lambda T'' + \frac{4\pi}{3k} B''(x) = -2\pi k \left\{ (1-R)\left[I_n - B(0)\right] + \frac{2}{3k} B'(0) \right\} E_2(kx) - 2\pi B'(0) E_3(kx). \tag{17}$$

Usually equations describing such processes do not contain exponential integrals.

The total thermal flux at a considerable distance from the boundary surface we set equal to zero. Then

$$T' \to 0 \quad \text{as} \quad x \to \infty. \tag{18}$$

The molecular heat transfer through the boundary is governed by the following boundary condition of the third kind

$$\lambda T' = \alpha (T - T_0) \text{ when } x = 0. \tag{19}$$

The solution to Eq. (17) with the boundary conditions (18) and (19) is

$$\lambda T(x) + \frac{4\sigma}{3k} T^4(x) = -\frac{2(1-R)}{k}\left[\pi I_n - \sigma T^4(0)\right] E_4(kx) + C, \tag{20}$$

where

$$C = \frac{\lambda}{\alpha} \cdot \frac{(1-R)\left[\pi I_n - \sigma T^4(0)\right]}{1 + (1-R)\frac{8\sigma}{3k\lambda} T^3(0)} + \frac{2\pi(1-R) I_n}{3k} + \frac{2\sigma(1+R) T^4(0)}{3k} + \lambda T_0.$$

The temperature gradient is

$$T'(x) = -\frac{2\left\{(1-R)\left[\pi I_n - \sigma T^4(0)\right] + \frac{8\sigma}{3k} R T^3(0) T'(0)\right\} E_3(kx) + \frac{8\sigma}{3k} T^3(0) T'(0) E_4(kx)}{\lambda + \frac{16\sigma}{3k} T^3(x)}, \tag{21}$$

where

$$T'(0) = \frac{(1-R)\left[\pi I_n - \sigma T^4(0)\right]}{\lambda + (1-R)\frac{8\sigma}{3k} T^3(0)}.$$

The accuracy of these expressions is of the order $O(1/\lambda k^2)$.

The unknown surface temperature $T(0)$ is found as the solution to the algebraic equation

$$\left(1 + \frac{8\alpha}{3k\lambda}\right)\sigma T^4(0) - \frac{8\sigma}{3k\lambda}\alpha T_0 T^3(0) + \frac{\alpha}{1-R} T(0) - \left[\frac{\alpha}{1-R} T_0 + \pi I_n\right] = 0. \tag{22}$$

In Fig. 2 are shown temperature fields in a solid medium calculated on a computer for the following values of the governing parameters:

$$\sigma = 5.67 \cdot 10^{-8}\ \text{W/m}^2 \cdot {}^\circ\text{K}; \quad \alpha = 10\ \text{W/m}^2 \cdot {}^\circ\text{C}; \quad R = 0.5;$$

$$I_n = 1300\ \text{W/m}^2; \quad T_0 = 300\ {}^\circ\text{K}.$$

A decrease in the product λk results in a lower surface temperature of the body, which in turn reduces the convective thermal flux transferred to the adjoining medium. The radiative component of the thermal flux at the body surface, which has been stipulated in the problem to be equal to the convective component, behaves analogously.

Expression (16) indicates that a decrease in the radiation flux is related to an increase in B'(0), which is proportional to the temperature gradient at the boundary and, therefore, the other component B(0) decreases with decreasing surface temperature. This is confirmed in Fig. 2.

We note, in conclusion, that, for analyzing the emissivity of bodies, their boundary with the surrounding medium must be characterized only in terms of its reflectivity and transmittivity with respect to radiation. Radiation is generated in the body within the boundary layer, where the radiation intensity is determined according to expressions (8) and (9).

In optically very dense media, where heat is transmitted only by conduction, the radiative component of the thermal flux is appreciable within the boundary layer. The heat transmission characteristics here are strongly affected by the boundary conditions with respect to radiation and the molecular flux.

NOTATION

$T(x)$	is the temperature;
T_0	is the temperature of adjoining medium;
I_n	is the intensity of external radiation;
I^+, I^-	are the radiation intensities along the positive and the negative x-axis;
$m = \cos\theta$;	
$B(x) = (\sigma/\pi)T^4$	is the emissivity of black body;
k	is the radiation absorptivity;
λ	is the thermal conductivity;
α	is the heat transfer coefficient;
R	is the surface reflectivity;
σ	is the Stefan–Boltzmann constant.

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